

# Von Neumann Modules, Intertwiners and Self-Duality\*

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## Abstract

We apply the ideas of Muhly, Skeide and Solel [MSS03] of considering von Neumann  $\mathcal{B}$ -modules as intertwiner spaces for representations of  $\mathcal{B}'$  to obtain a new, simple and self-contained proof for self-duality of von Neumann modules. This simplifies also the approach of [MSS03].

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# 1 Introduction

Let  $E$  be a Hilbert module over a von Neumann algebra  $\mathcal{B} \subset \mathcal{B}(G)$  acting (non-degenerately) on the Hilbert space  $G$ . We define the Hilbert space  $H = E \odot G$  as the interior **tensor product over**  $\mathcal{B}$  of the right  $\mathcal{B}$ -module  $E$  and the  $\mathcal{B}$ - $\mathbb{C}$ -module  $G$  with inner product  $\langle x_1 \odot g_1, x_2 \odot g_2 \rangle = \langle g_1, \langle x_1, x_2 \rangle g_2 \rangle$ . Every  $x \in E$  gives rise to a mapping  $L_x: g \mapsto x \odot g$  in  $\mathcal{B}(G, H)$  and it is easy to verify that  $L_x b = L_x b$  and  $L_x^* L_y = \langle x, y \rangle$ .

We, therefore, may and will identify every Hilbert  $\mathcal{B}$ -module over a von Neumann algebra  $\mathcal{B} \subset \mathcal{B}(G)$  as a concrete submodule  $E \subset \mathcal{B}(G, H)$  of operators, where  $H = E \odot G$ . Following Skeide [Ske00, Ske01] we say  $E$  is a **von Neumann  $\mathcal{B}$ -module**, if  $E$  is strongly closed in  $\mathcal{B}(G, H)$ .

On  $H$  we define a (normal unital) representation  $\rho': \mathcal{B}' \rightarrow \mathcal{B}(H)$  of the commutant  $\mathcal{B}'$  of  $\mathcal{B}$  by  $\rho'(b')(id_E \odot b')$ . (This is well-defined, because  $b'$  is a bilinear mapping on the  $\mathcal{B}$ - $\mathbb{C}$ -module  $G$ , and also checking normality is routine.) In the special case when  $E$  is the GNS-module of a completely positive mapping with values in  $\mathcal{B}$  (see Paschke [Pas73]),  $\rho'$  is known as *commutant lifting* (Arveson [Arv69]).

Following Skeide [Ske98, Ske00], the  $\mathcal{B}'$ -**center** of the  $\mathcal{B}'$ - $\mathcal{B}$ -module  $\mathcal{B}(G, H)$  is defined as

$$C_{\mathcal{B}'}(\mathcal{B}(G, H)) = \{x \in \mathcal{B}(G, H) : \rho'(b')x = xb' \ (b' \in \mathcal{B}')\}.$$

As observed, for instance, by Goswami and Sinha [GS99], it is easy to check that  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is a itself a von Neumann  $\mathcal{B}$ -module.

Clearly,  $E$  is contained in  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ .

It is the starting point in [MSS03] to show that  $E$  is all of  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ . Once known that von Neumann modules are **self dual**, i.e. every bounded right linear mapping  $\Phi: E \rightarrow \mathcal{B}$  (so-called  **$\mathcal{B}$ -functionals**) has the form  $\langle x, \bullet \rangle$  for a (unique)  $x \in E$ , (see [Ske00, Ske01] for proof using *complete quasi orthonormal systems*, a suitable generalization of orthonormal bases in Hilbert spaces) this task is easy: Like for Hilbert spaces a strongly closed (and, therefore, self-dual) submodule with zero-complement is all. And since  $EG$  is total in  $H$  the complement of  $E$  in  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is, indeed,  $\{0\}$ .

**1.1 Remark.** The other important observation in [MSS03] is that for an arbitrary (normal unital) representation  $\rho'$  of  $\mathcal{B}'$  on a Hilbert space,  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is a von Neumann  $\mathcal{B}$ -module acting totally on  $G$ , what gives a one-to-one correspondence between von Neumann  $\mathcal{B}$ -modules contained in  $\mathcal{B}(G, H)$  and representations  $\rho'$  of  $\mathcal{B}'$  on  $H$ .

This approach becomes particularly fruitful, when the von Neumann modules are two-sided so that there is around another (normal unital) representation  $\rho$  on  $H$  of a second von Neumann algebra  $\mathcal{A}$ . Switching the roles of  $\mathcal{B}$  and  $\mathcal{A}'$ , the result is a duality between  $\mathcal{A}$ - $\mathcal{B}$ -modules and

$\mathcal{B}'$ – $\mathcal{A}'$ –modules generalizing the duality between a von Neumann algebra and its commutant. One application is a complete theory of normal representations of the adjointable operators on a von Neumann  $\mathcal{B}$ –module on a von Neumann  $\mathcal{A}$ –module (this can be, e.g., a Hilbert space).

In this short note we show that  $E = C_{\mathcal{B}'}(\mathcal{B}(G, H))$  and we show that  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is self-dual, thus, showing that  $E$  is self-dual. The only prerequisite for the first statement is von Neumann’s *double commutant theorem*, the only prerequisite for the second statement is a technical lemma which asserts that every  $\mathcal{B}$ –functional  $\Phi$  is represented by an operator in  $\mathcal{B}(H, G)$  (see below). The simplicity of the proofs improves also accessibility of [MSS03] and, therefore, of the whole theory of von Neumann modules.

## 2 $E = C_{\mathcal{B}'}(\mathcal{B}(G, H))$

Every  $a \in \mathcal{B}^a(E)$  (the **algebra of adjointable operators** on  $E$ ) gives rise to a bounded operator  $x \odot g \mapsto ax \odot g$  on  $H$ . In that way, we identify  $\mathcal{B}^a(E)$  as a  $*$ –subalgebra of  $\mathcal{B}(H)$ . It is easy to see that  $\mathcal{B}^a(E)$  is a von Neumann subalgebra of  $\mathcal{B}(H)$ .

It follows that the *matrix  $*$ –algebra*

$$\mathcal{M} = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$$

with the obvious operations is a von Neumann algebra on  $G \oplus H$ . Let us compute its commutant.

**2.1 Proposition.** *The commutant of  $\mathcal{M}$  is  $\mathcal{M}' = \left\{ \begin{pmatrix} b' & 0 \\ 0 & \rho'(b') \end{pmatrix} : b' \in \mathcal{B}' \right\}$ .*

PROOF. Suppose  $\begin{pmatrix} b' & y'^* \\ x' & a' \end{pmatrix} \in \mathcal{B}(G \oplus H)$  is an element in  $\mathcal{M}'$ . As it must commute with  $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$  ( $b \in \mathcal{B}$ ) we find

$$\begin{pmatrix} b'b & 0 \\ x'b & 0 \end{pmatrix} = \begin{pmatrix} bb' & by'^* \\ 0 & 0 \end{pmatrix}.$$

As this must hold for all  $b \in \mathcal{B}$  (in particular also for  $b = \mathbf{1}$ ), we find  $x' = y' = 0$  and  $b' \in \mathcal{B}'$ . The remaining part  $\begin{pmatrix} b' & 0 \\ 0 & a' \end{pmatrix}$  must commute with  $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$  ( $x \in E$ ). Therefore,

$$\begin{pmatrix} 0 & a'x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xb' \\ 0 & 0 \end{pmatrix}.$$

We find  $a'(x \odot g) = a'xg = xb'g = \rho'(b')(x \odot g)$  for all  $x \in E, g \in G$  and, therefore,  $a' = \rho'(b')$ . ■

The commutant of  $\mathcal{M}'$  is, clearly,

$$\mathcal{M}'' = \begin{pmatrix} \mathcal{B} & C_{\mathcal{B}'}(\mathcal{B}(H, G)) \\ C_{\mathcal{B}'}(\mathcal{B}(G, H)) & \rho'(\mathcal{B}')' \end{pmatrix}.$$

By the *double commutant theorem*  $\mathcal{M}'' = \mathcal{M}$ . Therefore, we do not only show the statement of this section's headline, but, as an additional benefit, we identify also  $\mathcal{B}^a(E)$  as the commutant of the image of  $\mathcal{B}'$  under  $\rho'$ . (This can also be done by using *Morita equivalence* for von Neumann algebras; see Rieffel [Rie74].)

**2.2 Proposition.**  $E = C_{\mathcal{B}'}(\mathcal{B}(G, H))$  and  $\mathcal{B}^a(E) = \rho'(\mathcal{B}')'$ .

### 3 $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is self-dual

A  $\mathcal{B}$ -functional  $\Phi \in \mathcal{B}^r(C_{\mathcal{B}'}(\mathcal{B}(G, H)), \mathcal{B})$  gives rise to a linear mapping

$$L_\Phi: \text{span} C_{\mathcal{B}'}(\mathcal{B}(G, H))G \rightarrow G \quad L_\Phi(x \odot g) = (\Phi x)g.$$

The proof of the following lemma consists, essentially, in showing that for computing the operator norm of  $L_\Phi$  it is sufficient to take the supremum only over elementary tensors  $x \odot g$  ( $\|x\| \leq 1, \|g\| \leq 1$ ).

**3.1 Lemma.**  $\|L_\Phi\| = \|\Phi\|$ . Therefore,  $L_\Phi$  extends to a bounded operator in  $\mathcal{B}(H, G)$  identified with  $\Phi$ .

PROOF. (Sketch only. See [Ske00, Ske01] for details.) Suppose that there is a cyclic vector  $g_0 \in G$ , i.e.  $\mathcal{B}g_0$  is dense in  $G$ . (Otherwise, use a decomposition of  $G$  into subspaces  $G_\alpha$  cyclic for  $\mathcal{B}$  and take into account the facts, firstly, that also  $H$  decomposes accordingly into cyclic subspaces  $H_\alpha$  and, secondly, that the norm of an element in a direct sum of operator spaces  $\mathcal{B}(G_\alpha, H_\alpha)$  is just the supremum over the single norms.) Then every element in  $H$  can be approximated by those of the form  $h = x \odot g_0$ . Use polar decomposition  $x = x_0|x|$  of  $x$  and put  $g = |x|g_0$ . Then  $\|h\| = \|g\|$  because  $g \in |x|G$ . In particular, every unit vector in  $H$  can be approximated by  $x \odot g$  where  $x$  is a partial isometry in  $E$  and  $g$  is a unit vector in  $G$ . ■

**3.2 Proposition.**  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is self-dual.

PROOF. From  $\Phi\rho'(b')(x \odot g) = \Phi(x \odot b'g) = \Phi x b'g = b'\Phi x g = b'\Phi(x \odot g)$  we see that  $\Phi$  intertwines  $b'$  and  $\rho'(b')$ . Therefore the adjoint  $y = \Phi^*$  of  $\Phi$  is an element in  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  such that  $\Phi x = \langle y, x \rangle$  for all  $x \in C_{\mathcal{B}'}(\mathcal{B}(G, H))$ . ■

## 4 Synthesis

**4.1 Theorem.** Every von Neumann  $\mathcal{B}$ -module is self-dual.

PROOF.  $E = C_{\mathcal{B}'}(\mathcal{B}(G, H))$  and  $C_{\mathcal{B}'}(\mathcal{B}(G, H))$  is self-dual. ■

**4.2 Remark.** It seems that Lemma 3.1 forms always an essential part of the proofs of self-duality, which cannot be replaced by simpler arguments.

**4.3 Remark.** To be honest, we should mention that (unlike the approach by complete quasi orthonormal systems) the preceding arguments cannot be used to show the *Riesz representation theorem* (Hilbert spaces are self-dual), but, actually, reduce the statement about von Neumann modules to that about Hilbert spaces. An equivalent form of the *Riesz representation theorem* is that all bounded operators between Hilbert spaces have an adjoint. Without this, in the proof of Proposition 3.2 it was not possible to pass from  $\Phi$  to  $\Phi^*$ . One may see the failure of the argument clearly, by taking  $\mathcal{B} = \mathbb{C}$  and  $G = \mathbb{C}$  and for  $H$  only a pre-Hilbert space. This is, actually, the only place in these notes, where we are not able to write down an adjoint explicitly on the algebraic domain  $\text{span } EG$ . (The adjoint of  $x: g \mapsto x \odot g$  is, of course,  $x^*: y \odot g \mapsto \langle x, y \rangle g$ .)

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